Exactly solvable spin ladder model with degenerate ferromagnetic and singlet states

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Abstract. We study the spin ladder model with interactions between spins on neighboring rungs. The model Hamiltonian with the exact singlet ground state degenerated with ferromagnetic state is obtained. The singlet ground state wave function has a special recurrent form and depends on two parameters. Spin correlations in the singlet ground state show double-spiral structure with period of spirals equals to the system size. For special values of parameters they have exponential decay. The spectrum of the model is gapless and there are asymptotically degenerated excited states for special values of parameters in the thermodynamic limit.

PACS. 75.10.Jm Quantized spin models

1 Introduction

There has been growing interest lately in quantum spin systems with frustrated interactions [1]. Of special importance are models for which it is possible to construct an exact ground state. The first example of such a model is the well-known Majumdar-Ghosh model [2] which is the $s = 1/2$ chain with antiferromagnetic interactions J_1 and J_2 of nearest neighbor and next-nearest neighbor spins, where $J_2 = J_1/2$. Afterwards, a large class of 1D models with exact ground state has been found and studied [3–10].

A considerable progress has been achieved in a construction of such models by using so-called matrix-product (MP) form of the ground state wave function [11,12]. One of the example of the MP state is the ground state of the Affleck–Kennedy–Lieb–Tasaki (AKLT) model [3], which is the generic model of the Haldane phase of $S = 1$ chain. However, the MP wave functions have typically short-ranged correlations and, as a rule, describe systems with the gapped spectrum.

In [13,14] we have studied the model with exact singlet ground state degenerate with ferromagnetic state. This model has two different nearest neighbor and next-nearest neighbor interactions depending on the parameter ν and

is described by the Hamiltonian

$$H = -\sum_{i=1}^{M} \left(\mathbf{S}_{2i-1} \cdot \mathbf{S}_{2i} - \frac{1}{4} \right)$$
$$- (\nu - 1) \sum_{i=1}^{M} \left(\mathbf{S}_{2i} \cdot \mathbf{S}_{2i+1} - \frac{1}{4} \right)$$
$$+ \frac{\nu - 1}{2\nu} \sum_{i=1}^{N} \left(\mathbf{S}_i \cdot \mathbf{S}_{i+2} - \frac{1}{4} \right) \tag{1}$$

with periodic boundary conditions and even $N = 2M$.

In fact, this Hamiltonian describe the line of transition points from the ferromagnetic to the singlet state of the model

$$H = -\sum_{i=1}^{M} \left(\mathbf{S}_{2i-1} \mathbf{S}_{2i} - \frac{1}{4} \right) + J_{23} \sum_{i=1}^{M} \left(\mathbf{S}_{2i} \mathbf{S}_{2i+1} - \frac{1}{4} \right)$$
$$+ J_{13} \sum_{i=1}^{N} \left(\mathbf{S}_i \mathbf{S}_{i+2} - \frac{1}{4} \right). \tag{2}$$

The ground state of (2) is ferromagnetic (singlet) at $\delta < 0$ ($\delta > 0$), where $\delta = J_{13} + \frac{J_{23}}{2(1 - J_{23})}$. When $\delta = 0$, the model (2) reduces to the Hamiltonian (1).

It has been proved in [13,14] that singlet ground state of (1) has zero energy at $\nu > 0$ as well as the ferromagnetic state, while all other states have the positive energies. It has also been shown that singlet ground state has double-spiral ordering (excluding some special values of parameter ν).

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In [13] we have also considered the second model which is equivalent to special case of the spin-$\frac{1}{2}$ ladder. This model depends on one parameter, has the non-degenerate singlet ground state and its ground state properties are similar to that of AKLT model.

The exact singlet ground state wave function of these models has a special recurrent form:

$$\Psi_0(M) = P_0\Psi_M, \tag{3}$$

$$\Psi_M =$$
$$(s_1^+ + \nu_1 s_2^+ + \nu_2 s_3^+ ... + \nu_2 s_N^+)(s_3^+ + \nu_1 s_4^+ ... + \nu_2 s_N^+)...$$
$$\times (s_{2n-1}^+ + \nu_1 s_{2n}^+ ... + \nu_2 s_N^+)...(s_{N-1}^+ + \nu_1 s_N^+) |\downarrow\downarrow ... \downarrow\rangle \tag{4}$$

where s_i^+ is the $s = \frac{1}{2}$ raising operator. Equation (4) contains $M = \frac{N}{2}$ operator multipliers and the vacuum state $|\downarrow\downarrow ... \downarrow\rangle$ is the state with all spins pointing down. The function Ψ_M is the eigenfunction of S_z with $S_z = 0$ but it is not the eigenfunction of $\mathbf{S}^2$. P_0 is a projector onto the singlet state.

The models in [13] correspond to two particular cases of this wave function: $\nu_1 = \nu_2 = \nu$ for the first model (1), and $\nu_1 + \nu_2 + 1 = 0$ for the second model.

In this paper we consider the general case of the wave function (3) which allows us to construct new Hamiltonian depending on two parameters ν_1 and ν_2 and having the degenerate singlet and ferromagnetic states. The singlet ground state of this Hamiltonian has double-spiral structure with period of spirals equals to the system size, but for special values of the parameters (which include the second model in [13]) the singlet ground state correlations have antiferromagnetic character with an exponential decay and the singlet wave function in these cases reduces to the MP form. The interesting feature of the model is the existence of the excited states which are asymptotically degenerated with the ground state for special values of parameters in the thermodynamic limit.

The paper is organized as follows. In the Section 2 we will consider the construction of the Hamiltonian with the exact singlet ground state and will calculate spin correlation functions. In the Section 3 the Hamiltonian at special values of the parameters is considered. In the Section 4 the numerical results for the energy spectrum of the model are presented and Section 5 gives a brief summary.

2 The model

Now we will construct the Hamiltonian for which $\Psi_0(M)$ is the exact ground state wave function. This Hamiltonian describes two-leg $s = \frac{1}{2}$ ladder with periodic boundary conditions (Fig. 1) and can be represented in a form

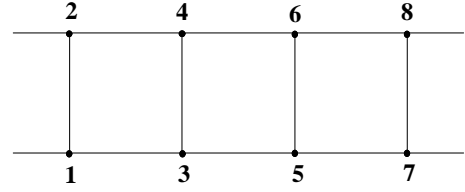$$H = \sum_{i=1}^{M} h_{i,i+1}, \tag{5}$$

Fig. 1. The two-leg spin ladder.

where $h_{i,i+1}$ describes the interaction between neighboring rungs. The spin space of two neighboring rungs consists of six multiplets: two singlet, three triplet and one quintet. At the same time, one can check that for open chain the wave function Ψ_M contains only three of the six multiplets of each pair of neighboring rungs: one singlet, one triplet and one quintet. The specific form of the singlet and triplet components present in the wave function (4) depends on parameters ν_1 and ν_2. The Hamiltonian $h_{i,i+1}$ can be written as the sum of the projectors onto the three missing multiplets with arbitrary positive coefficients $\lambda_1, \lambda_2, \lambda_3$:

$$h_{i,i+1} = \sum_{k=1}^{3} \lambda_k P_k^{i,i+1}, \tag{6}$$

where $P_k^{i,i+1}$ is the projector onto the missing multiplets in the corresponding cell Hamiltonian. Actually, the Hamiltonian $h_{i,i+1}$ contains also offdiagonal projectors between two missing triplets, but we use this freedom in advance to make exchange integrals on two legs of ladder and on each rung respectively equal, that is $J_{12} = J_{34}$ and $J_{13} = J_{24}$.

The wave function (4) is an exact wave function of the ground state of the Hamiltonian $h_{i,i+1}$ with zero energy, because

$$h_{i,i+1}|\Psi_M\rangle = 0, \qquad i = 1, ...M - 1 \tag{7}$$

and $\lambda_1, \lambda_2, \lambda_3$ are the excitation energies of the corresponding multiplets.

So, Ψ_M is the exact ground state wave function with zero energy for the total Hamiltonian of an open ladder

$$H_{op} = \sum_{i=1}^{M-1} h_{i,i+1} \tag{8}$$

$$H_{op}|\Psi_M\rangle = 0. \tag{9}$$

Since the function Ψ_M contains components with all possible values of total spin S ($0 \leq S \leq M$), then the ground state of open ladder is multiple degenerate. But it can be proved by the same way as it was made in [13,14] that for a cyclic ladder (5) only singlet and ferromagnetic components of Ψ_M have zero energy. Therefore, for a cyclic ladder (5) $\Psi_0(M)$ is a singlet ground state wave function degenerated with ferromagnetic state. Besides, the following general statements are valid for the model (5):

1) the ground states of open ladder described by (8) in the sector with fixed total spin S are non-degenerate and their energies are zero;

2) for cyclic ladder the ground state in the $S = 0$ sector is non-degenerate. The ground state energies for $0 < S < M$ are positive;
3) the singlet ground state wave functions for open and cyclic ladders coincide with each other.

Since the specific form of the existing and missing multiplets in the wave function (4) on each two nearest neighbor spin pairs depends on the parameters ν_1 and ν_2, the projectors in (6) also depend on ν_1 and ν_2. Each projector can be written in the form

$$
\begin{aligned}
P_k^{1,2} &= J_{12}^{(k)}(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_3 \cdot \mathbf{S}_4) + J_{13}^{(k)}(\mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_4) \\
&+ J_{14}^{(k)}\mathbf{S}_1 \cdot \mathbf{S}_4 + J_{23}^{(k)}\mathbf{S}_2 \cdot \mathbf{S}_3 + J_1^{(k)}(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_3 \cdot \mathbf{S}_4) \\
&+ J_2^{(k)}(\mathbf{S}_1 \cdot \mathbf{S}_3)(\mathbf{S}_2 \cdot \mathbf{S}_4) + J_3^{(k)}(\mathbf{S}_1 \cdot \mathbf{S}_4)(\mathbf{S}_2 \cdot \mathbf{S}_3) \\
&+ C^{(k)}
\end{aligned}
$$

and this representation is unique for a fixed value of the parameters ν_1 and ν_2.

Substituting the above expressions for the projectors into equation (6), we obtain the general form of the Hamiltonians $h_{i,i+1}$. Inasmuch as the Hamiltonians $h_{i,i+1}$ have exactly the same form for any i, it suffices here to give the expression for $h_{1,2}$:

$$
\begin{aligned}
h_{1,2} &= J_{12}(A_{12} + A_{34}) \\
&+ J_{13}(A_{13} + A_{24}) + J_{14}A_{14} + J_{23}A_{23} \\
&+ J_1 A_{12}A_{34} + J_2 A_{13}A_{24} + J_3 A_{14}A_{23} \qquad (10)
\end{aligned}
$$

where

$$
A_{ij} = \mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4}
$$

and all exchange integrals depend on the model parameters and the spectrum of excited states $J_i = J_i(\nu_1, \nu_2, \lambda_1, \lambda_2, \lambda_3)$ as follows:

$$
J_{12} = -\frac{\lambda_2}{2} + \frac{\lambda_3}{2}\frac{(\nu_1 - 1)^2 - \nu_2^2}{(\nu_1 - 1)^2 + \nu_2^2}
$$

$$
J_{13} = -\frac{\lambda_2}{2} - \frac{\lambda_3}{2}\frac{(\nu_1 - 1)^2 - \nu_2^2}{(\nu_1 - 1)^2 + \nu_2^2}
$$

$$
J_{14} = -\frac{\lambda_2}{2} - \frac{\lambda_3}{2}\frac{(\nu_2 - \nu_1 + 1)^2}{(\nu_1 - 1)^2 + \nu_2^2}
$$

$$
J_{23} = -\frac{\lambda_2}{2} - \frac{\lambda_3}{2}\frac{(\nu_2 + \nu_1 - 1)^2}{(\nu_1 - 1)^2 + \nu_2^2}
$$

$$
J_1 = 2J_{12} - \lambda_1\frac{(2\nu_1 - \nu_2\nu_1 - \nu_2)(\nu_2\nu_1 + \nu_2 - 1 - \nu_1^2)}{Z}
$$

$$
J_2 = 2J_{13} - \lambda_1\frac{(\nu_1 - 1)^2(\nu_2\nu_1 + \nu_2 - 1 - \nu_1^2)}{Z}
$$

$$
J_3 = J_{14} + J_{23} - \lambda_1\frac{(\nu_1 - 1)^2(2\nu_1 - \nu_2\nu_1 - \nu_2)}{Z} \qquad (11)
$$

where

$$
Z = \frac{3}{4}(\nu_1 - 1)^4 + \frac{1}{4}(\nu_1 + 1)^2(2\nu_2 - \nu_1 - 1)^2
$$

(one should keep in mind that only positive λ_i can be substituted to these expressions).

In general, the Hamiltonian $h_{i,i+1}$ contains all the terms presented in (10), but we can simplify it by setting, for example, $J_2 = J_3 = 0$ and solving equations (11) for λ_1, λ_2, λ_3. All λ_i turn out to be positive in this case for any ν_1 and ν_2 except two lines: $\nu_1 = 1$ and $\nu_2 = \nu_1 + 1$, where ground state is multiple degenerated. The Hamiltonian $h_{i,i+1}$ in this case takes the form

$$
\begin{aligned}
h_{1,2} &= J_{12}(A_{12} + A_{34}) + J_{13}(A_{13} + A_{24}) \\
&+ J_{14}A_{14} + J_{23}A_{23} + J_1 A_{12}A_{34} \qquad (12)
\end{aligned}
$$

$$
J_{12} = \frac{\nu_1\nu_2 + \nu_2 - 2\nu_1 - \nu_2^2}{2}
$$

$$
J_{13} = \frac{\nu_1\nu_2 + \nu_2 - \nu_1^2 - 1}{2}
$$

$$
J_{14} = \nu_1 - \nu_2 \qquad J_{23} = \nu_1(1 - \nu_2)
$$

$$
J_1 = 4\frac{\nu_1(1 - \nu_2)(\nu_1 - \nu_2)}{(1 - \nu_1)^2} \, .
$$

The calculation of the norm of (3) and the singlet ground state correlation functions can be performed in complete analogy to the corresponding calculations for the case $\nu_1 = \nu_2$ [13,14]. So, a norm of $\Psi_0(M)$ can be written in a form

$$
G_M = \langle \Psi_0(M)\Psi_0(M) \rangle = \frac{1}{2}\int_{-1}^{1} \Phi_M(y)\mathrm{d}y, \qquad (13)
$$

where $\Phi_M(y)$ is expanded over Legendre polynomials $P_n(y)$

$$
\Phi_M(y) = \sum_{n=0}^{M} c_n(M)P_n(y). \qquad (14)
$$

The coefficients $c_n(l)$ are defined by the recurrent equation

$$
\begin{aligned}
c_n(l + 1) &= \frac{n}{2n - 1}\frac{[\nu_2(n - 1) + \nu_1 + 1]^2}{2}c_{n-1}(l) \\
&+ \frac{\nu_2^2(n^2 + n) + (\nu_1 - 1)^2}{2}c_n(l) \\
&+ \frac{n + 1}{2n + 3}\frac{[\nu_2(n + 2) - \nu_1 - 1]^2}{2}c_{n+1}(l) \qquad (15)
\end{aligned}
$$

with initial condition $c_0(0) = 1$ and $c_n(l) = 0$ at $n > l$.

The appropriate calculations result in the expression for spin correlation functions at $N \to \infty$

$$
\langle \mathbf{S}_1 \mathbf{S}_{2l+1} \rangle = \langle \mathbf{S}_2 \mathbf{S}_{2l+2} \rangle = \frac{1}{4}\cos\left(\frac{4\pi l}{N}\right) \qquad (16)
$$

$$
\langle \mathbf{S}_1 \mathbf{S}_{2l+2} \rangle = \frac{1}{4}\cos\left(\frac{4\pi l}{N} + \triangle\varphi\right). \qquad (17)
$$

These equations mean that the spiral on each leg with pitch angle $\frac{4\pi}{N}$ is formed and the shift angle between spirals on the upper and the lower legs is $\triangle\varphi = \frac{4\pi}{N}\frac{\nu_1 - 1}{\nu_2}$. So, there is just one full rotation of the spin over the length of the ladder, independent of the size of the system and for fixed $l \ll N$ at $N \to \infty$ two spins on the ladder are parallel.

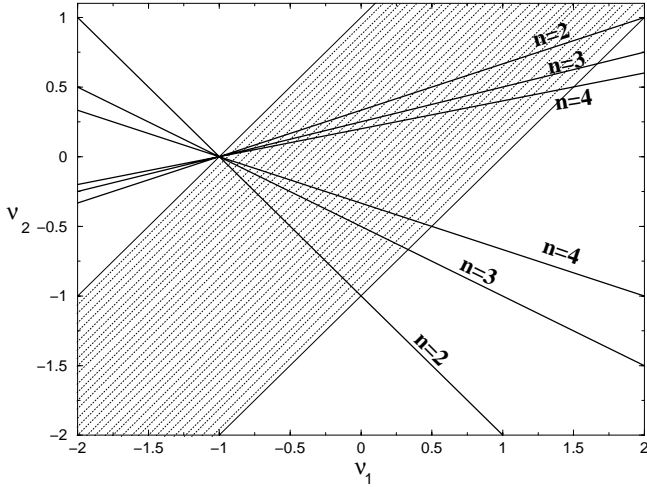**Fig. 2.** (ν_1, ν_2) plane. Thick lines are special cases of the model. Each value of n corresponds to a pair of equivalent lines. Due to the symmetry (21) it is sufficient to consider only the shaded area of (ν_1, ν_2) plane.

3 Special cases

There are special values of the parameters ν_1 and ν_2 for which equations (16, 17) are not valid. These values can be determined from equation (15), when the coefficients of $c_{n-1}(l)$ or $c_{n+1}(l)$ equal to zero. These conditions lead to following equations

$$\nu_2(n-1) + \nu_1 + 1 = 0 \tag{18}$$
$$-\nu_2(n+1) + \nu_1 + 1 = 0. \tag{19}$$

The special lines (18) and (19) on the (ν_1, ν_2) plane are shown in Figure 2. At $\nu_1 = \nu_2$ equations (18, 19) give special points of the model (1) [13,14].

We note that there is a symmetry transformation

$$\nu_1 \rightarrow \frac{\nu_2 - 1}{\nu_2 - \nu_1} \quad \nu_2 \rightarrow \frac{\nu_2}{\nu_2 - \nu_1} \tag{20}$$

which maps lines (18) and (19) onto each other and, therefore, there is pair of equivalent lines for each value of n. The transformation (20) does not change the wave function (3) as well as the Hamiltonian (12) apart from a change in energy scale by a factor $(\nu_2 - \nu_1)^2$. So, it is sufficient to consider the region on (ν_1, ν_2) plane restricted by the inequality $|\nu_2 - \nu_1| \leq 1$ (see Fig. 2).

For ν_1 and ν_2 defined by equations (18) or (19) $\Phi_M(y)$ contains only n terms and the wave function Ψ_M contains only n multiplets rather than $(M+1)$ as it does in generic case. It can be shown [13] that wave function $\Psi_0(M)$ for these special cases can be written in MP form:

$$\Psi_0(M) = \text{Tr}\,(D_{1,2}\,D_{3,4}\,\dots\,D_{N-1,N}), \tag{21}$$

where $D = T + uS$ is the $n \times n$ matrix describing states of corresponding spin pair. Singlet state matrix is
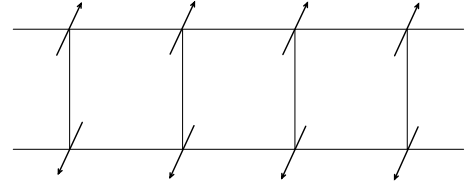
$$S = I\,|s\rangle \tag{22}$$

Fig. 3. Stripe spin structure on the ladder model.

where I is identity matrix and $|s\rangle$ is the singlet state. Triplet state matrix T is expressed by Clebsch-Gordan coefficients $C_{m_1,m_2} = \langle (1, m_1)\,(j, m_2) \mid (j, m_1 + m_2)\rangle$ as follows:

$$T = \frac{1}{C_{0,j}} \begin{pmatrix} C_{0,j}\,|0\rangle & C_{1,j-1}\,|1\rangle & 0 & 0 & 0 \\ C_{-1,j}\,|-1\rangle & C_{0,j-1}\,|0\rangle & . & 0 & 0 \\ 0 & . & . & . & 0 \\ 0 & 0 & . & . & C_{1,-j}\,|1\rangle \\ 0 & 0 & 0 & C_{-1,-j+1}\,|-1\rangle & C_{0,-j}\,|0\rangle \end{pmatrix}, \tag{23}$$

where $j = \frac{n-1}{2}$ and $|\sigma\rangle$ is the triplet state with $S^z = \sigma$. The parameter u is defined by expression

$$u = \frac{\nu_1 - 1}{\nu_2\,(n-1)}\,. \tag{24}$$

Exact calculation of the correlators in these cases using standard transfer matrix technique results in

$$\langle \mathbf{S}_1 \mathbf{S}_2 \rangle = \frac{1}{4} - \frac{u^2}{\omega_1}$$

$$\langle \mathbf{S}_i \mathbf{S}_{i+2l} \rangle = (1+2z)\frac{u^2 - z^2}{\omega_1^2}\left(\frac{\omega_2}{\omega_1}\right)^{l-1}$$

$$\langle \mathbf{S}_{2i+1} \mathbf{S}_{2i+2l+2} \rangle = -(1+2z)\frac{(u+z)^2}{\omega_1^2}\left(\frac{\omega_2}{\omega_1}\right)^{l-1}$$

$$\langle \mathbf{S}_{2i+2} \mathbf{S}_{2i+2l+1} \rangle = -(1+2z)\frac{(u-z)^2}{\omega_1^2}\left(\frac{\omega_2}{\omega_1}\right)^{l-1} \tag{25}$$

where we use notations

$$z = \frac{1}{n-1}, \quad \omega_1 = 1 + 2z + u^2, \quad \omega_2 = \omega_1 - 4z^2.$$

In the particular case of zero singlet weight, $u = 0$, when spins on each rung form a local triplet, correlation functions (25) coincide with those obtained in [11,16].

According to equations (25) the singlet ground state has collinear or stripe spin structure, *i.e.* spin-spin correlations are ferromagnetic along legs and antiferromagnetic between them (Fig. 3).

These correlations have an exponential decay for finite value n and the correlation length r_c is

$$r_c = 2\ln^{-1}\left|\frac{\omega_1}{\omega_2}\right|. \tag{26}$$

For $n \to \infty$ there is a magnetic order m:

$$\langle \mathbf{S}_i \mathbf{S}_{i+2l} \rangle = - \langle \mathbf{S}_i \mathbf{S}_{i+1+2l} \rangle = m^2 \qquad l \to \infty,$$

$$m = \frac{u}{1+u^2} .$$

When $u = \pm 1$ the magnetic order is equal to the classical value $1/2$.

It is interesting to note that r_c diverges for finite n when $u \to \infty$ $((\nu_1, \nu_2) \to (-1, 0))$ but the prefactors in (25) vanish in this case. As it is shown in Figure 2 this is the point where all special lines are intersected and the wave function $\Psi_0(M)$ is a simple product of the singlet pairs

$$\Psi_0(M) = (s_1^+ - s_2^+) \dots (s_{N-1}^+ - s_N^+) |\downarrow\downarrow \dots \downarrow\rangle. \qquad (27)$$

We note that the wave function (3) show double-spiral ordering for all values of ν_1 and ν_2 excluding the special lines. The crossover between spiral and strip states occurs in the exponentially small (at $N \to \infty$) vicinity of the special lines.

Now we should make the following remark. There is one particular case $\nu_1 + \nu_2 + 1 = 0$ $(n = 2)$, which was considered in [13], when on each two neighboring rungs the wave function Ψ_M contains only one singlet and one triplet and does not contain quintet. Therefore, in this case the cell Hamiltonian can be written in the form [15]

$$H_{i,i+1} = \sum_{k=1}^{4} \lambda_k P_k^{i,i+1}, \qquad (28)$$

where $P_4^{i,i+1}$ - is a projector onto the quintet state. If $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$, wave function $\Psi_0(M)$ is non-degenerate singlet ground state wave function for the Hamiltonian (28). In this case all four-spin interactions can be excluded by setting $J_1 = J_2 = J_3 = 0$ and then we arrive at the Hamiltonian

$$H_{i,i+1} = J_{12} \left[\left(\mathbf{S}_{2i-1} \cdot \mathbf{S}_{2i} - \frac{1}{4} \right) + \left(\mathbf{S}_{2i+1} \cdot \mathbf{S}_{2i+2} - \frac{1}{4} \right) \right]$$
$$+ 2J_{13} \left[\left(\mathbf{S}_{2i-1} \cdot \mathbf{S}_{2i+1} - \frac{1}{4} \right) + \left(\mathbf{S}_{2i} \cdot \mathbf{S}_{2i+2} - \frac{1}{4} \right) \right]$$
$$+ 2J_{14}(\mathbf{S}_{2i-1} \cdot \mathbf{S}_{2i+2} - \frac{1}{4}) + 2J_{23}(\mathbf{S}_{2i} \cdot \mathbf{S}_{2i+1} - \frac{1}{4}) \qquad (29)$$

and all exchange integrals J_{ij} depend on one model parameter. The explicit form of J_{ij} was found in [13]. It was shown that for this model $\Psi_0(M)$ is non-degenerate singlet ground state wave function with exponentially decaying spin correlations and there is an energy gap.

In other special cases we can also construct Hamiltonians for which $\Psi_0(M)$ is non-degenerate singlet ground state wave function. But we have to introduce more distant interactions. For example, such a model for $n = 3$ would contain interactions between next-nearest neighbor spin pairs

$$H = \sum_{i=1}^{M} H_{i,i+1,i+2}. \qquad (30)$$

For the point $(\nu_1 = 1, \nu_2 = -1)$, when spins on each rung form a local triplet, one of the possible Hamiltonians can be written as

$$H_{1,2,3} = A_{12} + A_{23} + 38A_{13} + 6A_{13}^2$$
$$- (A_{12}^2 + A_{23}^2)(A_{13} + \frac{7}{4})$$
$$- (A_{13} + \frac{7}{4})(A_{12}^2 + A_{23}^2) + 52 \qquad (31)$$

where

$$A_{ij} = \mathbf{L}_i \cdot \mathbf{L}_j - 1$$

and $\mathbf{L}_i = \mathbf{s}_{2i-1} + \mathbf{s}_{2i}$ is the $S = 1$ operator.

The ground state wave function for the Hamiltonian (31) has the matrix-product form (21) with

$$D = \begin{pmatrix} |0\rangle & -|1\rangle & 0 \\ |-1\rangle & 0 & -|1\rangle \\ 0 & |-1\rangle & -|0\rangle \end{pmatrix} .$$

4 Spectrum of the model

Generally, the excitation spectrum of the model (5, 12) can not be calculated exactly. It is clear that this spectrum is gapless because, for example, the one-magnon energy is $\sim N^{-4}$ at $N \to \infty$. Moreover, the lowest singlet excitation is gapless as well. For the model (5, 12) lying on the special lines this fact can be established from the following consideration. For the simplicity we consider the case $\nu_1 = \nu_2 = \nu = \frac{1}{n}$. We choose the variational wave function of the excited singlet state at $\nu_1 = \nu_2 = \nu$ as

$$\Psi_s(\nu, \delta, M) = \frac{1}{\sqrt{1 - c^2(\delta, M)}}$$
$$\times [c(\delta, M) \, \Psi_0(\nu, M) - \Psi_0(\nu + \delta, M)],$$

where $\Psi_0(\nu, M)$ and $\Psi_0(\nu + \delta, M)$ are the normalized singlet ground state wave functions of (5) with zero energies at $\nu_1 = \nu_2 = \nu$ and $\nu_1 = \nu_2 = \nu + \delta$ (the point $\nu_1 = \nu_2 = \nu + \delta$ on (ν_1, ν_2) plane does not belong to the special line). The functions $\Psi_s(\nu, \delta, M)$ and $\Psi_0(\nu, M)$ are orthogonal and $c(\delta, M)$ is the overlap of $\Psi_0(\nu, M)$ and $\Psi_0(\nu + \delta, M)$

$$c(\delta, M) = \langle \Psi_0(\nu, M) \mid \Psi_0(\nu + \delta, M) \rangle .$$

It can be shown that

$$c^2(\delta, M) \sim \frac{1}{1 + \delta^2 (M!)^2 e^{O(M)}} \quad at \quad M \to \infty. \qquad (32)$$

Equation (32) follows from the fact that $\langle \Psi_0(\nu, M) \mid \Psi_0(\nu, M) \rangle$ and $c(\delta, M)$ is represented by the sum of n terms in (14) while $\langle \Psi_0(\nu + \delta, M) \mid \Psi_0(\nu + \delta, M) \rangle$ contains M terms.
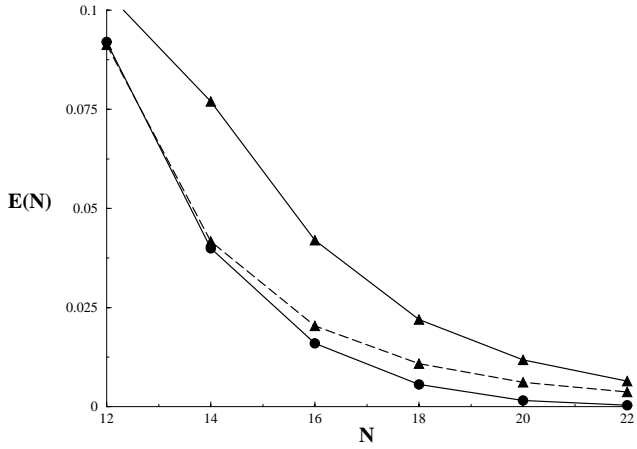
Fig. 4. Dependences of energy of the lowest singlet (circles) and triplet (triangles) excited states on the system size N at the points $\nu_1 = \nu_2 = 1/2$ (solid lines) and $\nu_1 = \nu_2 = 3/4$ (dashed line). The singlet excited state energy for $\nu_1 = \nu_2 = 3/4$ is out of scale of this figure but has similar dependence on N.

The variational energy calculated with respect to $\Psi_s(\nu, \delta, M)$ is

$$
\begin{aligned}
E_s &= \langle \Psi_s(\nu, \delta, M) \mid H(\nu, M) \mid \Psi_s(\nu, \delta, M) \rangle \\
&= \frac{1}{1 - c^2(\delta, M)} \\
&\quad \times \langle \Psi_0(\nu + \delta, M) \mid H(\nu, M) \mid \Psi_0(\nu + \delta, M) \rangle \\
&\sim \frac{1 + \delta^2 (M!)^2}{\delta^2 (M!)^2} \\
&\quad \times \left\langle \Psi_0(\nu + \delta, M) \mid \left(-\delta \frac{\mathrm{d}H(\nu, M)}{\mathrm{d}\nu}\right) \mid \Psi_0(\nu + \delta, M) \right\rangle \\
&\sim \frac{1 + \delta^2 (M!)^2}{\delta (M!)^2} .
\end{aligned}
$$

And minimization of E_s over the function $\delta(M)$ leads to

$$
\delta \sim \mathrm{e}^{-M \ln M + O(M)} \qquad E_s \sim \mathrm{e}^{-M \ln M + O(M)} .
$$

This consideration can be easily extended to the points $\nu_1 \neq \nu_2$ on the special lines. But it is not valid for the parameters ν_1 and ν_2 which are out of special lines. We performed the numerical diagonalization of finite ladders for various parameters ν_1 and ν_2. The energies of lowest singlet and triplet states of (5) are shown in Figure 4 as a function of N for parameters corresponding to different types of the ground state. Figure 4 shows that the exponential degeneracy possibly takes place for all parameters ν_1 and ν_2, but we can not confirm it strictly.

Thus, on the special lines the ground state of considered model is asymptotically degenerated at the thermodynamic limit. It is not clear if the degeneracy is exponentially large or not.

So far, we have considered the models with degenerate singlet and ferromagnetic states. Now we discuss the phase diagram of the zigzag chain model given by (2). The line of transition points from the ferromagnetic to singlet state
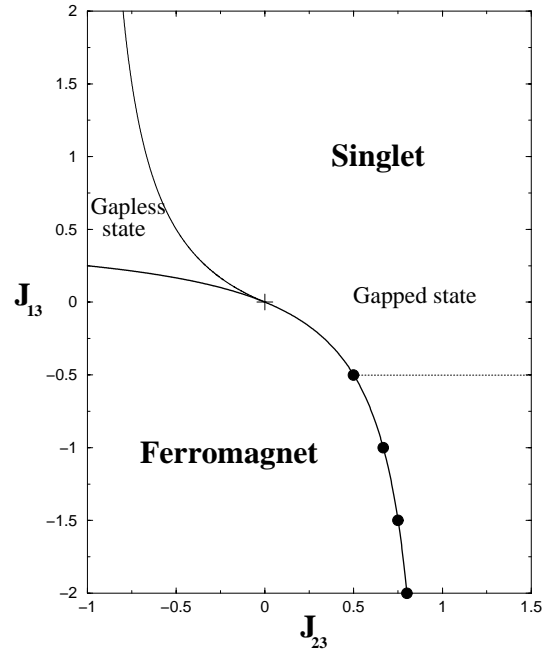

Fig. 5. Phase diagram of the zigzag chain model (2). The thick solid line is the boundary between the ferromagnetic and singlet phases. Circles correspond to the special cases of the model. The thin solid line denotes the heuristic boundary between gapped and gapless phases. On the dotted line the ground state is a product of singlet pairs [2, 3][4, 5]....

is described by the one parametric Hamiltonian (1) (or by (12) at $\nu_1 = \nu_2$).

The exact ground state in the singlet phase (Fig. 5) is generally unknown. But it is interesting to note that the ground state on the line $J_{13} = -1/2$ is the product of singlets on ladder diagonals (2,3), (4,5), ... as in the point $J_{13} = -J_{23} = -1/2$ on the transition line. The spectrum of (2) on the transition line is gapless. There are some regions on the plane (J_{13}, J_{23}) which were studied by different approximation methods.

At $J_{13} = 0$ and $0 < J_{23} < 1$ the model (2) reduces to the alternating Heisenberg chain studied in [17]. The lowest excitation is the triplet and there is the gap. At $J_{23} = 0$ and $J_{13} > 0$ the model (2) reduces to the spin ladder with antiferromagnetic interactions along legs and the ferromagnetic interactions on rungs. It is evident that there is a gap at $J_{13} \ll 1$ (in this case the model is equivalent to the spin $S = 1$ Heisenberg chain). It was shown in [18] that the gap exists at $J_{13} \gg 1$. At $J_{23} = -1$ and $J_{13} \gg 1$ the spectrum is gapless according to the results of [19].

We have calculated the first singlet and triplet excitation at $J_{13} = -1/2$ and $1/2 < J_{23} < 1$ by the numerical diagonalization of the finite ladders. As it can be seen from Figure 6 the gap is closed on the transition line. So, we expect that the phase diagram of the model (2) has the form shown in Figure 5.
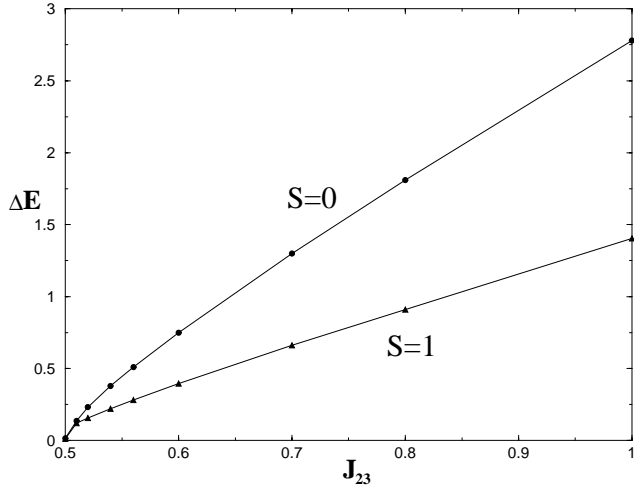
Fig. 6. Dependence of singlet-singlet and singlet-triplet energy gap on J_{23} along the line $J_{13} = -1/2$ (dotted line in Fig. 5). Calculation was made for finite chain with $N = 20$.

5 Summary

We have constructed the spin ladder model with ferro- and antiferromagnetic interactions between spins on neighboring rungs. The model has exact singlet ground state degenerated with ferromagnetic state. The spin correlators in the singlet ground state show double-spiral ordering with period of spirals equals to the system size. However, for special values of the parameters spin correlators in the singlet state have exponential decay and in these cases the singlet ground state wave function can be represented in the MP form. The spectrum of the model is gapless and there is asymptotic degeneracy of the ground state for special values of the parameters at the thermodynamic limit.

The singlet ground state wave function has the recurrent form (3) and depends on two parameters. This function can be further generalized. For example, we can take $\Psi_0(M)$ in a form

$$\Psi_0(M) = P_0 \Psi_M$$

where Ψ_M is the product of alternating multipliers

$$\Psi_M = (s_1^+ + \nu_1 s_2^+ + \nu_2 s_3^+ ...)(s_3^+ + \nu_3 s_4^+ + \nu_4 s_5^+ ...)$$
$$\times (s_5^+ + \nu_1 s_6^+ + \nu_2 s_7^+ ...)... |\!\downarrow\downarrow ... \downarrow\rangle$$

with the condition

$$\frac{1 + \nu_1}{\nu_2} = \frac{1 + \nu_3}{\nu_4} .$$

The Hamiltonian of the model for which $\Psi_0(M)$ is the singlet ground state has two rungs in the elementary cell. This Hamiltonian can be taken in the form containing the interactions between neighboring rungs only and without four spin terms.

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